To show: the group of real numbers under addition ($\mathbb{R}$,+) is isomorphic to the group of complex numbers under addition ($\mathbb{C}$,+).

Consider the real numbers as a vector space over the rationals. A theorem that follows from the Axiom of Choice says that every vector space has a basis; thus this vector space does. Call such a basis **B**.

To review what it means for **B** to be a basis of this vector space: the elements of **B** are linearly independent, and **B** spans $\mathbb{R}$, i.e. any real number x can be expressed as a finite linear combination of elements in **B**:

$$x = q_1\mathbf{v}_1 + q_2\mathbf{v}_2 + ... + q_n\mathbf{v}_n$$

where each $q_i \in \mathbb{Q}$ and each $\mathbf{v}_i \in \mathbf{B}$. For any given x, there's only one such linear combination expressing it (this follows from the linear independence).

Consider the set of all ordered pairs $(\mathbf{v}, 0)$ or $(0, \mathbf{v})$ where $\mathbf{v} \in \mathbf{B}$; call this set **BB**. That is, $\mathbf{BB} = (\mathbf{B} \times \{0\}) \cup (\{0\} \times \mathbf{B})$. ($\times$ is Cartesian product)

With each ordered pair $(\mathbf{v}, 0)$ interpreted as the real number $\mathbf{v}$ and each ordered pair $(0, \mathbf{v})$ interpreted as $\mathbf{v}i$, **BB** is a basis for the vector space of the complex numbers over the rationals.

B and **BB** are both infinite sets with the same cardinality (the cardinality of $\mathbb{R}$). Therefore there exists a 1-to-1 correspondence between **B** and **BB**. Any 1-to-1 correspondence will do for our purpose.

A 1-to-1 correspondence between **B** and **BB** creates a linear map and thus an isomorphism between their respective vector spaces. If this step is not obviously true, please trust that there's an established theorem saying that if two vector spaces over the same field have the same dimension, they are isomorphic with respect to all vector space operations, one of which is vector addition. (The dimension of a vector space is the cardinality of any basis for it.) Thus the group ($\mathbb{R}$,+) is isomorphic to the group ($\mathbb{C}$,+), Q.E.D.